Exercise 59

Find all points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

Solution

The tangent line to $f(x)$ is horizontal wherever the first derivative is zero. Evaluate the first derivative.

$$\begin{aligned} f'(x) = \frac{df}{dx} &= \frac{d}{dx}(2\sin x + \sin^2 x) \\ &= 2\frac{d}{dx}(\sin x) + \frac{d}{dx}(\sin^2 x) \\ &= 2(\cos x) + (2\sin x)\cdot\frac{d}{dx}(\sin x) \\ &= 2(\cos x) + (2\sin x)\cdot(\cos x) \\ &= 2\cos x + 2\sin x\cos x \\ &= 2\cos x(1+\sin x) \end{aligned}$$

Set it equal to zero.

$$2\cos x(1+\sin x) = 0$$

Solve for x.

$$\begin{aligned} &2\cos x = 0 && \text{or} && 1+\sin x = 0 \\ &\cos x = 0 && \text{or} && \sin x = -1 \\ &x = \frac{1}{2}(2n-1)\pi, \quad n = 0, \pm 1, \pm 2, \ldots && \text{or} && x = \frac{3\pi}{2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Find the corresponding y-values by plugging these values of x into the function for $f(x)$.

$$\begin{aligned} f\left(\frac{1}{2}(2n-1)\pi\right) &= 2\sin\left[\frac{1}{2}(2n-1)\pi\right] + \sin^2\left[\frac{1}{2}(2n-1)\pi\right] = 2(-1)^{n+1} + 1 \\ f\left(\frac{3\pi}{2} + 2n\pi\right) &= 2\sin\left(\frac{3\pi}{2} + 2n\pi\right) + \sin^2\left(\frac{3\pi}{2} + 2n\pi\right) = 2(-1) + 1 = -1 \end{aligned}$$

Therefore, the points on the curve $f(x) = 2\sin x + \sin^2 x$ that have a horizontal tangent line are

$$\left(\frac{1}{2}(2n-1)\pi, 2(-1)^{n+1} + 1\right) \quad \text{and} \quad \left(\frac{3\pi}{2} + 2n\pi, -1\right),$$

where n is an integer.

This is confirmed in the graph of $f(x)$ versus x.